

GENERALIZED ZALCMAN CONJECTURE FOR CONVEX FUNCTIONS OF ORDER α

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ABSTRACT. Let \mathcal{S} denote the class of all functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$ which are analytic and univalent in the open unit disk \mathbb{D} and, for $\lambda > 0$, let $\Phi_\lambda(n, f) = \lambda a_n^2 - a_{2n-1}$ denote the generalized Zalcman coefficient functional. Zalcman conjectured that if $f \in \mathcal{S}$, then $|\Phi_1(n, f)| \leq (n-1)^2$ for $n \geq 3$. The functional of the form $\Phi_\lambda(n, f)$ is indeed related to Fekete-Szegő functional of the n -th root transform of the corresponding function in \mathcal{S} . This conjecture has been verified for a certain special geometric subclasses of \mathcal{S} but the conjecture remains open for $f \in \mathcal{S}$ and for $n > 6$. In the present paper, we prove sharp bounds on $|\Phi_\lambda(n, f)|$ for $f \in \mathcal{F}(\alpha)$ and for all $n \geq 3$, in the case that λ is a positive real parameter, where $\mathcal{F}(\alpha)$ denotes the family of all functions $f \in \mathcal{S}$ satisfying the condition

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad \text{for } z \in \mathbb{D},$$

where $-1/2 \leq \alpha < 1$. Thus, the present article proves the generalized Zalcman conjecture for convex functions of order α , $\alpha \in [-1/2, 1)$.

1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{H} be the set of all analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A} = \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}$. Clearly each $f \in \mathcal{A}$ has the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

For a constant $\alpha \in [-1/2, 1)$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{F}(\alpha)$ if f satisfies the condition

$$(2) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad \text{for } z \in \mathbb{D}.$$

A number of important properties of the family $\mathcal{F}(\alpha)$ for various special values of α may be obtained from the literature [6, 9, 10, 21]. For example, the family $\mathcal{F}(0)$ consists of normalized convex functions, usually denoted by the symbol \mathcal{K} , and thus, for $\alpha \in [0, 1)$, functions in $\mathcal{F}(\alpha)$ are convex in \mathbb{D} . Moreover, a function $f \in \mathcal{A}$ is convex precisely when the function $g(z) = zf'(z)$ is starlike, i.e. $g(\mathbb{D})$ is a domain which is starlike (with respect to the origin). Thus, $g \in \mathcal{A}$ is starlike if $\operatorname{Re}(zg'(z)/g(z)) > 0$ in \mathbb{D} . Also, it is worth

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to recall that functions in $\mathcal{F}(-1/2)$ (and hence in $\mathcal{F}(\alpha)$ for $\alpha \in [-1/2, 0)$) are known to be convex in one direction (and hence functions in $\mathcal{F}(-1/2)$ are close-to-convex and univalent in \mathbb{D}) but are not necessarily starlike in \mathbb{D} (see [24]). Moreover, if $f \in \mathcal{F}(\alpha)$ and is of the form (1), then one has the following necessary coefficient inequality (see for instance, [23, Theorem 5.6, p.324])

$$(3) \quad |a_n| \leq A_n := \frac{\Gamma(n+1-2\alpha)}{n!\Gamma(2-2\alpha)}$$

for $\alpha \in [-1/2, 1)$ and for all $n \geq 2$, where $\Gamma(\cdot)$ denotes the usual gamma function. See the relation (8) in the proof of Lemma 1 for a quick proof of the coefficient inequality (3). Throughout $A_n := A_n(\alpha)$ denotes the Taylor coefficients of the extremal function $f_\alpha \in \mathcal{F}(\alpha)$, where

$$(4) \quad f_\alpha(z) = \frac{1 - (1-z)^{2\alpha-1}}{2\alpha-1}$$

and for $\alpha = 1/2$ this is interpreted as the limiting case which gives $f_{1/2}(z) = -\log(1-z)$. We refer to the recent articles [3, 22] for certain properties of *sections/partial sums* of functions from the class $\mathcal{F}(-1/2)$. The importance of the class $\mathcal{F}(-1/2)$ connected with certain univalent harmonic mappings are considered in [5].

The aim of this article is to solve the generalized Zalcman coefficient conjecture for the class $\mathcal{F}(\alpha)$. We begin to present necessary preliminaries.

One of the classical problems is to find for each $\lambda > 0$ the maximum modulus value of the *generalized Zalcman functional*

$$\Phi_\lambda(n, f) := \lambda a_n^2 - a_{2n-1}$$

over the class \mathcal{S} of functions f of the form (1). First we remark that the functional for the case $n = 2$ is fundamental in the investigation of a number of problems in function theory and is popularly known as Fekete and Szegő functional [8]. Secondly, we observe that for $\theta \in \mathbb{R}$,

$$\Phi_\lambda(n, e^{-i\theta} f(e^{i\theta} z)) = e^{2(n-1)i\theta} \Phi_\lambda(n, f(z))$$

and thus, $|\Phi_\lambda(n, f)|$ is invariant under rotation. For the special case $\lambda = 1$, $\Phi_1(n, f)$ is simply referred to as the *Zalcman functional* for $f \in \mathcal{S}$. In 1960, Lawrence Zalcman conjectured that the sharp inequality

$$(5) \quad |\Phi_1(n, f)| = |a_n^2 - a_{2n-1}| \leq (n-1)^2$$

holds for $f \in \mathcal{S}$ and for all $n \geq 3$, with equality only for the Koebe function $k(z) = z/(1-z)^2$ and its rotations. This remarkable conjecture, also called the Zalcman coefficient inequality, which was posed as an approach to prove the Bieberbach conjecture, was investigated by many mathematicians, and remained open for all $n > 6$.

By means of Loewner's method, Fekete and Szegő [8] (see also [6, Theorem 3.8]) indeed obtained the following result. Later in 1960, the same was derived by Jenkins [11] by means of his general coefficient theorem.

Theorem A. ([8]) *For each $f \in \mathcal{S}$, $|\lambda a_n^2 - a_{2n-1}| \leq 1 + 2 \exp(-2\lambda(1-\lambda))$ for $0 \leq \lambda < 1$. The bound is sharp for each λ .*

In particular, for $\lambda \rightarrow 1^-$, we have the well-known Fekete-Szegő inequality (i.e. the case $n = 2$ of (5)):

$$|\Phi_1(2, f)| = |a_2^2 - a_3| \leq 1$$

(see also [21, Theorem 1.5]). In 1985, Pfluger [19] employed the variational method to present another treatment of Fekete-Szegő inequality and in 1986, Pfluger [20] used the method of Jenkins to obtain Theorem A for certain complex values of λ . Yet another important remark is that the functional $\Phi_1(2, f) = a_2^2 - a_3$ becomes $S_f(0)/6$, where S_f denotes the Schwarzian derivative which is defined for locally univalent function f by

$$S_f = (f''/f')' - (1/2)(f''/f')^2.$$

Next, if we consider the n -th root transform

$$g(z) = \sqrt[n]{f(z^n)} := z \sqrt[n]{f(z^n)/z^n} = z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + \dots$$

of $f \in \mathcal{S}$ with the power series of (1), we find that

$$\lambda a_2^2 - a_3 = n(\mu c_{n+1}^2 - c_{2n+1}),$$

where $\mu = \lambda n + (n - 1)/2$. This observation clearly defines the role of the generalized Zalcman functional in the class \mathcal{S} through the Fekete-Szegő functional. Thus, generalized Zalcman functional can be regarded as the generalization of Fekete-Szegő functional defined as in Theorem A.

Sharp bound for the generalized Fekete-Szegő functional has been established for several subclasses of \mathcal{S} (see [4, 16, 17]) and more recently in [1, 14, 15]. The Zalcman coefficient inequality for $n = 3$ and for the full class \mathcal{S} , was established in [12] and also for the special cases $n = 4, 5, 6$ in [13]. Recently, the authors in [1] considered the generalized Fekete-Szegő inequality for $\mathcal{F}(\alpha)$ and the generalized Zalcman coefficient inequality for the class $\mathcal{F}(-1/2)$. We refer to Theorems 2.1, 2.2 and 3.3 in [14] for the precise formulation of these results.

In this note we solve the generalized Zalcman coefficient inequality for the class $\mathcal{F}(\alpha)$ and obtain certain earlier known results as corollaries to it (see, for example, Corollary 1). In Section 2, we present a number of lemmas and the main results are stated and proved in Section 3.

2. REPRESENTATION OF FUNCTIONS IN $\mathcal{F}(\alpha)$

Lemma 1. *Let $-\frac{1}{2} \leq \alpha < 1$ and $f \in \mathcal{F}(\alpha)$ as in the form (1). Then we have*

$$(6) \quad f'(z) = \int_0^{2\pi} (1 - e^{i\theta} z)^{2\alpha-2} d\nu(\theta),$$

where $\nu(\theta)$ is a probability measure on $[0, 2\pi]$ and for $n \geq 2$,

$$(7) \quad |\lambda a_n^2 - a_{2n-1}| \leq (\lambda A_n^2 - 2A_{2n-1}) \int_0^{2\pi} \cos^2(n-1)\theta d\nu(\theta) + A_{2n-1},$$

where $A_n := A_n(\alpha)$ is given by (3).

Proof. Let $f \in \mathcal{F}(\alpha)$ for some $\alpha \in [-1/2, 1)$. By the well-known Herglotz representation theorem for analytic functions p with positive real part in \mathbb{D} , $p(0) = 1$, and the analytic characterization of $f \in \mathcal{F}(\alpha)$ given by (2), one has

$$\frac{1}{1-\alpha} \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) = p(z) := \int_0^{2\pi} \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} d\nu(\theta), \quad |z| < 1,$$

where $\nu(\theta)$ is a probability measure on $[0, 2\pi]$. By a computation, we easily have

$$\begin{aligned} f'(z) &= \int_0^{2\pi} (1 - e^{i\theta}z)^{2\alpha-2} d\nu(\theta) \\ &= 1 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2\alpha-2)(2\alpha-3)\cdots(2\alpha-n)}{(n-1)!} \left(\int_0^{2\pi} e^{i(n-1)\theta} d\nu(\theta) \right) z^{n-1}. \end{aligned}$$

By comparing the coefficients of z^{n-1} on both sides of the above equation, we easily have

$$(8) \quad a_n = A_n \int_0^{2\pi} e^{i(n-1)\theta} d\nu(\theta) \quad \text{for } n = 2, 3, \dots,$$

where $A_n = A_n(\alpha)$ is given by (3), i.e.

$$A_n = \frac{\Gamma(n+1-2\alpha)}{n!\Gamma(2-2\alpha)}, \quad n \geq 2.$$

We observe that the last relation quickly gives the necessary coefficient inequality (3) for functions in $\mathcal{F}(\alpha)$. Since $|\lambda a_n^2 - a_{2n-1}|$ is invariant under rotations, we can consider instead the problem of maximizing the functional $\operatorname{Re}(\lambda a_n^2 - a_{2n-1})$. Consequently, by (8), we begin to observe that

$$\begin{aligned} \operatorname{Re}(\lambda a_n^2 - a_{2n-1}) &= \lambda A_n^2 \left(\int_0^{2\pi} \cos(n-1)\theta d\nu(\theta) \right)^2 - \lambda A_n^2 \left(\int_0^{2\pi} \sin(n-1)\theta d\nu(\theta) \right)^2 \\ &\quad - A_{2n-1} \int_0^{2\pi} \cos 2(n-1)\theta d\nu(\theta). \end{aligned}$$

If we apply Cauchy-Schwarz inequality to the first integral above and use the trigonometric identity $\cos 2t = 2\cos^2 t - 1$ in the third integral, we find that

$$\begin{aligned} \operatorname{Re}(\lambda a_n^2 - a_{2n-1}) &\leq \lambda A_n^2 \int_0^{2\pi} \cos^2(n-1)\theta d\nu(\theta) - 2A_{2n-1} \int_0^{2\pi} \cos^2(n-1)\theta d\nu(\theta) + A_{2n-1} \\ &= (\lambda A_n^2 - 2A_{2n-1}) \int_0^{2\pi} \cos^2(n-1)\theta d\nu(\theta) + A_{2n-1}. \end{aligned}$$

The proof is completed. \square

Here is an alternate approach to Lemma 1 which works for a more general setting. If X is a linear topological space, then a subset Y of X is called convex if $tx + (1-t)y \in Y$ whenever $x, y \in Y$ and $0 \leq t \leq 1$. The closed convex hull of Y is defined as the intersection of all closed convex sets containing Y . A point $u \in Y$ is called an extremal point of Y if $u = tx + (1-t)y$, $0 < t < 1$ and $x, y \in Y$, implies that $x = y$. See [10, 18] for a general reference and for many important results on this topic.

In order to solve the generalized Zalcman coefficient inequality problem for the class $\mathcal{F}(\alpha)$, we need the following lemma.

Lemma B. ([10]) *Suppose that $F_\alpha(z) = (1-z)^{2\alpha-2}$ and $\alpha \in [-1/2, 1/2]$. If $s(F_\alpha)$, $\mathcal{H}s(F_\alpha)$ and $\mathcal{E}\mathcal{H}s(F_\alpha)$ denote the set of analytic functions subordinate to F_α , the closed convex hull of $s(F_\alpha)$ and the set of the extremal points of $\mathcal{H}s(F_\alpha)$, respectively, then $\mathcal{H}s(F_\alpha)$ consists of all analytic functions represented by*

$$(9) \quad F_\alpha(z) = \int_{|x|=1} (1-xz)^{2\alpha-2} d\mu(x),$$

where $\mu(x)$ is a probability measure on the unit circle $\partial\mathbb{D}$. Moreover, $\mathcal{E}\mathcal{H}s(F_\alpha)$ consists of the functions given by

$$(10) \quad F_\alpha(z) = (1-xz)^{2\alpha-2},$$

where $|x| = 1$.

If $\mathcal{F} \subset \mathcal{H}$ is convex and $L : \mathcal{H} \rightarrow \mathbb{R}$ is a real-valued functional on \mathcal{A} , then we say that L is convex on \mathcal{F} provided that

$$L(tg_1 + (1-t)g_2) \leq tL(g_1) + (1-t)L(g_2)$$

whenever $g_1, g_2 \in \mathcal{F}$ and $0 \leq t \leq 1$.

Since $\mathcal{H}s(F_\alpha)$ is convex, we have a real-valued, continuous and convex functional on $\mathcal{H}s(F_\alpha)$.

Lemma 2. *Suppose that $g(z) = 1 + \sum_{n=2}^{\infty} b_n z^{n-1}$ is analytic in \mathbb{D} and*

$$J(g) = \lambda \frac{(\operatorname{Re} b_n)^2}{n^2} - \frac{\operatorname{Re} b_{2n-1}}{2n-1},$$

where $\lambda > 0$. Then J is a real-valued, continuous and convex functional on $\mathcal{H}s(F_\alpha)$.

Proof. Let $h(z) = 1 + \sum_{n=2}^{\infty} c_n z^{n-1}$ be analytic in \mathbb{D} and $0 \leq t \leq 1$. By the definition of J , we have

$$J(h) = \lambda \frac{(\operatorname{Re} c_n)^2}{n^2} - \frac{\operatorname{Re} c_{2n-1}}{2n-1}$$

and thus,

$$\begin{aligned} J(tg + (1-t)h) &= \lambda \frac{t^2 (\operatorname{Re} b_n)^2 + 2t(1-t)\operatorname{Re} b_n \operatorname{Re} c_n + (1-t)^2 (\operatorname{Re} c_n)^2}{n^2} \\ &\quad - \left(\frac{t\operatorname{Re} b_{2n-1} + (1-t)\operatorname{Re} c_{2n-1}}{2n-1} \right). \end{aligned}$$

Therefore, by rearrangements, we find that

$$J(tg + (1-t)h) - tJ(g) - (1-t)J(h) = -\frac{\lambda(t(1-t))}{n^2} (\operatorname{Re} b_n - \operatorname{Re} c_n)^2 \leq 0,$$

which implies that J is a real-valued, continuous and convex functional on $\mathcal{H}s(F_\alpha)$. \square

Since $s(F_\alpha)$ is compact, for J as defined in Lemma 2, Theorem 4.6 in [10] yields the following lemma.

Lemma 3. *We have*

$$\max\{J(f) : f \in \mathcal{H}s(F_\alpha)\} = \max\{J(f) : f \in s(F_\alpha)\} = \max\{J(f) : f \in \mathcal{E}\mathcal{H}s(F_\alpha)\}.$$

Lemma 4. *Let $-\frac{1}{2} \leq \alpha < \frac{1}{2}$, $\lambda > 0$ and $f \in \mathcal{F}(\alpha)$ be as in the form (1). Then f has the form (6) and for $n \geq 2$, we have*

$$(11) \quad |\lambda a_n^2 - a_{2n-1}| \leq (\lambda A_n^2 - 2A_{2n-1}) \cos^2(n-1)\theta + A_{2n-1},$$

where A_n is given by (3).

Proof. By Lemma 1, there exists a function g analytic in \mathbb{D} such that

$$g(z) = \int_0^{2\pi} (1 - e^{i\theta} z)^{2\alpha-2} d\nu(\theta) = 1 + \sum_{n=2}^{\infty} b_n z^{n-1},$$

and $f'(z) = g(z)$ so that $na_n = b_n$. Now, Lemma B shows that $f' = g \in \mathcal{H}s(F_\alpha)$.

Since $|\lambda a_n^2 - a_{2n-1}|$ is invariant under rotations, we can consider instead the problem of maximizing the functional $\operatorname{Re}(\lambda a_n^2 - a_{2n-1})$ so that

$$\begin{aligned} \operatorname{Re}(\lambda a_n^2 - a_{2n-1}) &= \lambda(\operatorname{Re} a_n)^2 - \lambda(\operatorname{Im} a_n)^2 - \operatorname{Re} a_{2n-1} \\ &\leq \lambda(\operatorname{Re} a_n)^2 - \operatorname{Re} a_{2n-1} \\ &= \lambda \frac{(\operatorname{Re} b_n)^2}{n^2} - \frac{\operatorname{Re} b_{2n-1}}{2n-1} = J(g). \end{aligned}$$

The above facts and Lemma 3 imply that

$$(12) \quad |\lambda a_n^2 - a_{2n-1}| \leq \max\{J(h) : h \in \mathcal{H}s(F_\alpha)\} = \max\{J(h) : h \in \mathcal{E}\mathcal{H}s(F_\alpha)\}.$$

Since each $h \in \mathcal{E}\mathcal{H}s(F_\alpha)$ is in the form

$$h(z) = (1 - e^{i\theta} z)^{2\alpha-2} = 1 + \sum_{n=2}^{\infty} n A_n e^{i(n-1)\theta} z^{n-1},$$

as in the proof of Lemma 1, the relation (12) reduces to

$$|\lambda a_n^2 - a_{2n-1}| \leq (\lambda A_n^2 - 2A_{2n-1}) \cos^2(n-1)\theta + A_{2n-1}$$

and the proof is complete. \square

Lemma 5. *For $-\frac{1}{2} \leq \alpha < 1$ and $n \geq 3$, we define*

$$(13) \quad C_n(\alpha) = \frac{2A_{2n-1}(\alpha)}{A_n^2(\alpha)},$$

where $A_n = A_n(\alpha)$ is given by (3). Then for fixed α , we have

- (1) $C_n(\alpha)$ is monotonically decreasing with respect to n and $C_n(\alpha) \leq C_3(\alpha)$ if $-\frac{1}{2} \leq \alpha < 0$;
- (2) $C_n(\alpha)$ is monotonically increasing with respect to n and $C_n(\alpha) \geq C_3(\alpha)$ if $0 < \alpha < 1$,

where

$$C_3(\alpha) = \frac{3(2\alpha-4)(2\alpha-5)}{5(2\alpha-2)(2\alpha-3)}.$$

Proof. Note that $C_n(0) = 2$ for all $n \geq 3$ and thus, there is nothing to prove for the case $\alpha = 0$. Clearly, $C_n(\alpha) > 0$ for $-\frac{1}{2} \leq \alpha < 1$. We only need to consider

$$\begin{aligned} \frac{C_{n+1}(\alpha)}{C_n(\alpha)} - 1 &= \frac{(n+1)^2(2n+1-2\alpha)(n-\alpha)}{n(2n+1)(n+1-2\alpha)^2} - 1 \\ &= \frac{\alpha[(n+1)(4n^2-n-1)-2\alpha(3n^2-1)]}{n(2n+1)(n+1-2\alpha)^2} \\ &= \frac{\alpha\varphi(n)}{n(2n+1)(n+1-2\alpha)^2}, \end{aligned}$$

where $\varphi(n) = 4n^3 + 3n^2(1-2\alpha) - 2n - 1 + 2\alpha$. Since $\varphi(n)$ is trivially an increasing function of n for $n \geq 3$ and for each $-\frac{1}{2} \leq \alpha < 1$, it follows that $\varphi(n) \geq \varphi(3) = 128 - 52\alpha > 0$. This observation shows that $C_{n+1}(\alpha) > C_n(\alpha)$ for $0 < \alpha < 1$ while $C_{n+1}(\alpha) < C_n(\alpha)$ if $-\frac{1}{2} \leq \alpha < 0$. The desired conclusion now follows. \square

3. MAIN RESULTS

Let $Co(\mathcal{F})$ denote the convex hull of the set \mathcal{F} and its closure by $\overline{Co}(\mathcal{F})$. In view of the extreme points method described in Lemma 4, our results continue to hold if we replace the assumption $f \in \mathcal{F}(\alpha)$ by $f \in \overline{Co}(\mathcal{F}(\alpha))$ for the case $-\frac{1}{2} \leq \alpha < \frac{1}{2}$. For example, we have the following results and for the sake of completeness we include the proofs here.

Theorem 1. *For $\mathcal{F}(0) := \mathcal{K}$, let $f \in \overline{Co}(\mathcal{K})$ as in the form (1). Then we have*

- (1) $|\lambda a_n^2 - a_{2n-1}| \leq \lambda - 1$ for $n \geq 3$ and $\lambda \geq 2$. The equality is attained for the function $f_0(z) = \frac{z}{1-z}$.
- (2) $|\lambda a_n^2 - a_{2n-1}| \leq 1$ for $n \geq 3$ and $0 < \lambda < 2$. The equality is attained by convex combination of convex functions in \mathcal{K} , namely, for the functions f in the form

$$f(z) = \sum_{k=0}^{2n-3} \alpha_k \frac{z}{1 - e^{i\theta_k} z},$$

where $0 \leq \alpha_k \leq 1$,

$$\theta_k = \frac{(2k+1)\pi}{2n-2} \quad \text{and} \quad \sum_{m=0}^{n-2} \alpha_{2m} = \sum_{m=0}^{n-2} \alpha_{2m+1} = \frac{1}{2}.$$

Proof. For $\alpha = 0$, we have $A_n = 1$ for all $n \geq 2$. Thus we can apply either Lemma 1 or Lemma 4. In either way, applying either (7) or (11) we see that for $\lambda \geq 2$, we have

$$|\lambda a_n^2 - a_{2n-1}| \leq (\lambda - 2) \cdot 1 + 1 = \lambda - 1,$$

where the equality is attained by the convex function $f_0(z) = \frac{z}{1-z}$. For $0 < \lambda < 2$, both (7) and (11) reduces to

$$|\lambda a_n^2 - a_{2n-1}| \leq 1,$$

which occurs when $\theta = \pi/(2(n-1))$ and at this point $\sin^2((n-1)\theta) = 1$. \square

Theorem 1(2) is also obtained recently in [7] (see also [2]). Next we consider the case $\alpha = -1/2$ and because of its independent interest we supply the proof.

Corollary 1. ([14, Theorem 3.3]) *Let $f \in \mathcal{F}(-1/2)$ and f be of the form (1). Then we have*

- (1) $|\lambda a_n^2 - a_{2n-1}| \leq \frac{(n+1)^2}{4}\lambda - n$ for $n \geq 3$ and $\lambda \geq \frac{3}{2}$. The equality is attained for the function $f_{-1/2}(z)$ given by

$$(14) \quad f_{-1/2}(z) = \frac{z - z^2/2}{(1-z)^2} = z + \sum_{n=2}^{\infty} \frac{1+n}{2} z^n.$$

- (2) $|\lambda a_n^2 - a_{2n-1}| \leq \frac{(n+1)^2}{4}\lambda - n$ for $0 < \lambda < \frac{3}{2}$ and $n > \frac{4-\lambda+2\sqrt{4-2\lambda}}{\lambda}$. The equality is attained for the function $f_{-1/2}(z)$ given by (14).

- (3) $|\lambda a_n^2 - a_{2n-1}| \leq n$ for $0 < \lambda < \frac{3}{2}$ and $3 \leq n \leq \frac{4-\lambda+2\sqrt{4-2\lambda}}{\lambda}$. The equality is attained for functions f in the following form

$$f(z) = \sum_{k=0}^{2n-3} \alpha_k \frac{2z - e^{i\theta_k} z^2}{2(1 - e^{i\theta_k} z)^2} = \sum_{k=0}^{2n-3} \alpha_k e^{-i\theta_k} f_{-1/2}(ze^{i\theta_k}),$$

where $0 \leq \alpha_k \leq 1$,

$$\theta_k = \frac{(2k+1)\pi}{2n-2} \quad \text{and} \quad \sum_{m=0}^{n-2} \alpha_{2m} = \sum_{m=0}^{n-2} \alpha_{2m+1} = \frac{1}{2}.$$

Proof. Set $\alpha = -1/2$ in Lemma 1 or Lemma 4, or apply Lemma 5 directly. Then, because $A_n(-1/2) = (n+1)/2$ for all $n \geq 2$, it is clear from (7) or (11) that

$$|\lambda a_n^2 - a_{2n-1}| \leq \begin{cases} \left(\lambda \frac{(n+1)^2}{4} - 2n \right) + n & \text{if } \lambda \geq \frac{8n}{(n+1)^2} \\ n & \text{if } 0 < \lambda \leq \frac{8n}{(n+1)^2}. \end{cases}$$

Clearly, $\frac{8n}{(n+1)^2} \leq \frac{3}{2}$ if and only if $(3n-1)(n-3) \geq 0$ and thus, the Case (1) follows. For $0 < \lambda < \frac{3}{2}$, we see that $\lambda \geq \frac{8n}{(n+1)^2}$ if and only if $\varphi(n) := \lambda n^2 - 2n(4-\lambda) + \lambda \geq 0$. Since

$$\varphi(n) = \lambda \left[n - \left(\frac{4-\lambda+2\sqrt{4-2\lambda}}{\lambda} \right) \right] \left[n - \left(\frac{4-\lambda-2\sqrt{4-2\lambda}}{\lambda} \right) \right],$$

Case (2) follows. Finally, in the last case the range of n shows that $\lambda < \frac{8n}{(n+1)^2}$ and thus, Case (3) is clear. \square

Setting $\lambda = 1$ and $\frac{3}{2}$ in Corollary 1, we obtain the following.

Corollary 2. *Let $f \in \mathcal{F}(-1/2)$ as in the form (1). Then we have*

- (1) $|a_n^2 - a_{2n-1}| \leq \frac{(n-1)^2}{4}$ for $n > 5$. The equality is attained for the function $f_{-1/2}(z)$ given by (14).
(2) $|a_n^2 - a_{2n-1}| \leq n$ for $3 \leq n \leq 5$.
(3) $\left| \frac{3}{2}a_n^2 - a_{2n-1} \right| \leq \frac{3n^2 - 2n + 3}{8}$ for $n > 3$. The equality is attained for the function $f_{-1/2}(z)$ given by (14).
(4) $\left| \frac{3}{2}a_3^2 - a_5 \right| \leq 3$.

Theorem 2. Let $n \geq 3$, $-\frac{1}{2} \leq \alpha < 0$, $f \in \mathcal{F}(\alpha)$ as in the form (1), $A_n = A_n(\alpha)$ and $C_n = C_n(\alpha)$ be given by (3) and (13), respectively.

(1) If $n \geq 3$ and $\lambda \geq C_3(\alpha)$, then we have

$$|\lambda a_n^2 - a_{2n-1}| \leq \lambda A_n^2 - A_{2n-1},$$

where the equality is attained for the function $f_\alpha(z)$ defined by (4).

(2) If $0 < \lambda < C_3(\alpha)$, then there exists a fixed $n_0 > 3$ such that

$$C_{n_0-1}(\alpha) > \lambda \geq C_{n_0}(\alpha).$$

If $0 < \lambda < C_3(\alpha)$ and $n \geq n_0$, then

$$|\lambda a_n^2 - a_{2n-1}| \leq \lambda A_n^2 - A_{2n-1},$$

where the equality is attained for the function $f_\alpha(z)$ defined by (4).

(3) If $0 < \lambda < C_3(\alpha)$ and $3 \leq n < n_0$, then

$$|\lambda a_n^2 - a_{2n-1}| \leq A_{2n-1},$$

where the equality is attained by convex combination of rotations of functions $f_\alpha \in \mathcal{F}(\alpha)$.

Proof. We apply Lemma 1 or Lemma 4. Thus, by (7) or (11), we find that

$$(15) \quad |\lambda a_n^2 - a_{2n-1}| \leq \begin{cases} \lambda A_n^2 - A_{2n-1} & \text{if } \lambda A_n^2 - 2A_{2n-1} \geq 0 \\ A_{2n-1} & \text{if } \lambda A_n^2 - 2A_{2n-1} \leq 0. \end{cases}$$

This is the key and using this and Lemma 5, we obtain the desired conclusion in each case. We remind that the inequality (15) holds for $-\frac{1}{2} \leq \alpha < 1$.

Case (1). $\lambda \geq C_3(\alpha)$ and $n \geq 3$.

In this case, $\lambda \geq C_3(\alpha) \geq C_n(\alpha)$ by Lemma 5 and thus, $\lambda A_n^2 - 2A_{2n-1} \geq 0$ for all $n \geq 3$, which implies the conclusion of Case (1), where the equality is attained by the function $f_\alpha(z)$.

If $0 < \lambda < C_3(\alpha)$, then there exists a fixed $n_0 > 3$ such that

$$C_{n_0-1}(\alpha) > \lambda \geq C_{n_0}(\alpha),$$

by Lemma 5. So we need to divide the case $0 < \lambda < C_3(\alpha)$ into the following two cases.

Case (2). $0 < \lambda < C_3(\alpha)$ and $n \geq n_0$.

In this case, Lemma 5 yields that $\lambda A_n^2 - 2A_{2n-1} \geq 0$ for $n \geq n_0$, and the conclusion follows from (15).

Case (3). $0 < \lambda < C_3(\alpha)$ and $3 \leq n < n_0$.

In this case, $\lambda A_n^2 - 2A_{2n-1} < 0$ for $3 \leq n < n_0$ and the desired inequality follows from (15). \square

Proof of the following theorem is similar and it just uses Lemma 5 and the equation (15). Thus, we include only the necessary details.

Theorem 3. Let $n \geq 3$, $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$ and $f \in \mathcal{F}(\alpha)$ as in the form (1), $A_n = A_n(\alpha)$ and $C_n = C_n(\alpha)$ be given by (3) and (13), respectively.

(1) If $n \geq 3$ and $0 < \lambda \leq C_3(\alpha)$, then

$$|\lambda a_n^2 - a_{2n-1}| \leq A_{2n-1},$$

where the equality is attained by convex combination of rotations of functions $f_\alpha \in \mathcal{F}(\alpha)$ defined by (4).

(2) If $\lambda > C_3(\alpha)$, then there exists a fixed $n_0 > 3$ such that

$$C_{n_0-1}(\alpha) < \lambda \leq C_{n_0}(\alpha).$$

Furthermore, if $\lambda > C_3(\alpha)$ and $n \geq n_0$, then

$$|\lambda a_n^2 - a_{2n-1}| \leq A_{2n-1},$$

where the equality is attained by convex combination of rotations of functions $f_\alpha \in \mathcal{F}(\alpha)$ defined by (4).

(3) If $\lambda > C_3(\alpha)$ and $3 \leq n < n_0$, then

$$|\lambda a_n^2 - a_{2n-1}| \leq \lambda A_n^2 - A_{2n-1},$$

where the equality is attained for the function $f_\alpha(z)$ defined by (4).

Proof. **Case (1).** $0 < \lambda \leq C_3(\alpha)$ and $n \geq 3$.

In this case, $\lambda \leq C_3(\alpha) \leq C_n(\alpha)$ by Lemma 5 and thus, $\lambda A_n^2 - 2A_{2n-1} \leq 0$, which by (15) implies the desired inequality

$$|\lambda a_n^2 - a_{2n-1}| \leq A_{2n-1}.$$

If $\lambda > C_3(\alpha)$, then there exists a fixed $n_0 > 3$ such that $C_{n_0-1}(\alpha) < \lambda \leq C_{n_0}(\alpha)$ by Lemma 5.

Case (2). $\lambda > C_3(\alpha)$ and $n \geq n_0$.

For this case, Lemma 5 yields that $\lambda A_n^2 - 2A_{2n-1} \leq 0$ and the conclusion follows from (15).

Case (3). $\lambda > C_3(\alpha)$ and $3 \leq n < n_0$.

In this case, $\lambda A_n^2 - 2A_{2n-1} \geq 0$ and the desired inequality follows as before. \square

Theorem 4. Let $f \in \mathcal{F}(1/2)$ as in the form (1). Then we have

- (1) $|\lambda a_n^2 - a_{2n-1}| \leq \frac{1}{2n-1}$ for $n \geq 3$ and $0 < \lambda \leq \frac{18}{5}$, where the equality is attained by convex combination of rotations of functions $f_{1/2} \in \mathcal{F}(1/2)$ defined by (4).
- (2) $|\lambda a_n^2 - a_{2n-1}| \leq \frac{1}{2n-1}$ for $\lambda > \frac{18}{5}$ and $n \geq \frac{\lambda + \sqrt{\lambda^2 - 2\lambda}}{2}$, where the equality is attained by convex combination of rotations of functions $f_{1/2} \in \mathcal{F}(1/2)$.
- (3) $|\lambda a_n^2 - a_{2n-1}| \leq \frac{\lambda}{n^2} - \frac{1}{2n-1}$ for $\lambda > \frac{18}{5}$ and $3 \leq n < \frac{\lambda + \sqrt{\lambda^2 - 2\lambda}}{2}$. The equality is attained for the function $f_{1/2}(z) = -\log(1-z)$.

Proof. By Lemma 1, we have

$$|\lambda a_n^2 - a_{2n-1}| \leq \left(\frac{\lambda}{n^2} - \frac{2}{2n-1} \right) \int_0^{2\pi} \cos^2(n-1)\theta \, d\nu(\theta) + \frac{1}{2n-1}.$$

If $n \geq 3$ and $0 < \lambda \leq \frac{18}{5}$, then

$$\frac{\lambda}{n^2} - \frac{2}{2n-1} \leq 0,$$

and the desired conclusion holds.

Similarly, if $\lambda > \frac{18}{5}$ and $n \geq \frac{\lambda + \sqrt{\lambda^2 - 2\lambda}}{2}$, then

$$\frac{\lambda}{n^2} - \frac{2}{2n-1} \leq 0,$$

which shows that the conclusion for this case holds.

If $\lambda > \frac{18}{5}$ and $3 \leq n < \frac{\lambda + \sqrt{\lambda^2 - 2\lambda}}{2}$, then

$$\frac{\lambda}{n^2} - \frac{2}{2n-1} > 0,$$

which provides a proof for this case. \square

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